NGO & SONS
ACADEMIC COACHING

2018 HSC Mathematics Extension 2 Solutions

Multiple Choice

Multiple Choice Answer Key

Question	Answer
1	B
2	C
3	D
4	C
5	A
6	A
7	D
8	B
9	B
10	C

Explanation

1. $$\begin{aligned}\int \frac{1}{\sqrt{1-4x^2}}\,dx &= \frac{1}{2}\int \frac{1}{\sqrt{\frac{1}{4}-x^2}}\,dx \\ &= \frac{1}{2}\sin^{-1} 2x + C\end{aligned}$$

Hence, the answer is B.

2. Rearranging the equation $9x^2 - 4y^2 = 36$ to $\dfrac{x^2}{4} - \dfrac{y^2}{9} = 1$ gives a value of $a = 2$ and $b = 3$.

Thus the asymptotes are $y = \pm\frac{b}{a}x = \pm\frac{3}{2}x$. Hence C.

3. Let $y = -\dfrac{1}{x} \implies x = -\dfrac{1}{y}$.

Then the equation with roots $-\frac{1}{\alpha}$, $-\frac{1}{\beta}$ and $-\frac{1}{\gamma}$ is

$$\begin{aligned}\left(-\frac{1}{y}\right)^3 + 2\left(-\frac{1}{y}\right)^2 + 5\left(-\frac{1}{y}\right) - 1 &= 0 \\ -1 + 2y - 5y^2 - y^3 &= 0\end{aligned}$$

A variable change from y back into x gives a cubic equation of $x^3 + 5x^2 - 2x + 1 = 0$. Hence D.

4.
- The square root term is positive, and hence y is positive. So the only possible answers are A and C.
- As $x \to \infty$, $y \to 0$, so the answer is C.

5. In this question we will use the annulus method. The radius of the solid is given by $y-(-1)=y+1=e^{3x}+1$. This immediately eliminates options C and D, meaning either A or B is correct. In the annulus method we have π on the outside of our integral (in the cylindrical shells method, we will have 2π on the outside of our integral). Hence, the answer is A.

6. Let $z^6=i$, where $z=r(\cos\theta+i\sin\theta)$.

Then $r^6(\cos\theta+i\sin\theta)^6=i$.
$r^6(\cos 6\theta+i\sin 6\theta)=\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}$.
Hence $r=1$ and $6\theta=\frac{\pi}{2}+k2\pi$ for some integer k.

$$\therefore \theta=\frac{\pi}{12}+k\frac{\pi}{3}$$

Looking through our options, both A and C satisfy the above condition. But the required modulus must also be 1. Hence, the answer is A.

7. The equation is equivalent to $\arg(z)=\arg(z-(-1+i))$ which is equivalent to $\arg(z)-\arg(z-(-1+i))=0$.

The angle between the vectors must be 0, and this occurs when they are pointing in the same direction. Hence the locus of z are the outer rays.

Hence D.

8. The function $F(x)=\displaystyle\int_0^x f(t)\,dt$ represents the *signed* area of $f(x)$ from 0 to x.

Consider the behaviour of $f(x)$ and $F(x)$ under these cases:

- From $0\to a$: The area under $f(x)$ increases at an increasing rate, and the curve $F(x)$ will be concave up.
- From $a\to b$: The area is increasing, though now at a decreasing rate. The curve $F(x)$ is now concave down.
- At $x=b$: There is a stationary point for $F(x)$. The area changes from increasing to decreasing.
- From $b\to c$: We now have negative area, causing $F(x)$ to decrease.
- At c: The area is still decreasing, but changes here from decreasing at an increasing rate, to decreasing at a decreasing rate. Hence there is a change in concavity for $F(x)$.
- From $c\to d$: The area decreases but more slowly. The curve is concave up.
- At d: The graph for $y=F(x)$ goes from decreasing to increasing. There is a stationary point here. The curve is still concave up.
- After the point d: The curve is still concave up. The area increases at an increasing rate.

Hence B.

9. This question makes use of two properties.

- A real number multiplied by a purely imaginary number is imaginary.
  Proof: Let the real number be a and let the imaginary number be bi. Then their product is abi which is purely imaginary.
- A purely imaginary number multiplied by a purely imaginary number is a real number.
  Proof: Let the two purely imaginary numbers be ai and bi, where a and b are real. Then their product is $abi^2=-ab$ which is purely real.

This question also requires the student to recognise the expressions can be converted into perfect squares.

$a^2p^2 + b^2q^2 - 2abpq = (ap - bq)^2$. Since the inside of the square is a purely imaginary number, its square is negative. Hence $(ap - bq)^2 < 0 \implies a^2p^2 + b^2q^2 < 2abpq$.

$a^2b^2 + p^2q^2 - 2abpq = (ab - pq)^2$. Since the inside of the square is a real number, its square is positive. Hence $(ap - bq)^2 > 0 \implies a^2p^2 + b^2q^2 > 2abpq$.

Hence B.

10. Consider $y = f(x)$ and $y = g(x)$. $f(x)$ is an odd function and $g(x)$ is an even function, meaning that the x-coordinates for their stationary points will be symmetric about the y-axis. Note that if we consider option A ($a < b$), we can pick values for a and b such that this will be true. However, as the functions' stationary point x-coordinates are symmetric about the y-axis, this means that $a < b$ implies that $b > a$, which is clearly a contradiction. Hence we cannot take A. Similarly, we can eliminate B as well. Now we have to pick either C or D. We know that one of C or D must be true so we will test a and b values. A stationary point of $f(x)$ is $x = a$, we will pick $a = \frac{\pi}{2}$. Now testing $b = \frac{\pi}{2}$, we get $g'(\frac{\pi}{2}) = 1$. Here we check if the stationary point is to the left or right of $x = a = \frac{\pi}{2}$. To do this, we will consider the gradient of $g'(x)$,

$$g''(x) = \cos x + \cos x - x\sin x = 2\cos x - x\sin x \Longrightarrow g''(\frac{\pi}{2}) = -\frac{\pi}{2}.$$

This means that our stationary point for $g(x)$ is to the right of our stationary point for $f(x)$, and hence,

$$|a| < |b|.$$

Therefore the answer is C.

Question 11

(a) (i)
$$\begin{aligned} zw &= (2+3i)(1-i) \\ &= 2-2i+3i+3 \\ &= 5+i. \end{aligned}$$

(ii)
$$\begin{aligned} \bar{z}-\frac{2}{w} &= 2-3i-2w^{-1} \\ &= 2-3i-(1+i) \\ &= 1-4i. \end{aligned}$$

(b) (i)
$$\begin{aligned} P(x) &= x^3+ax^2+b \\ P(4) &= 4^3+a\cdot 4^2+b \\ 0 &= 64+16a+b. \end{aligned}$$
$$\begin{aligned} P'(x) &= 3x^2+2ax \\ P'(4) &= 3\cdot 4^2+2a\cdot 4 \\ 0 &= 48+8a \\ a &= -6. \\ 0 &= 64+16(-6)+b \\ b &= 32 \end{aligned}$$

Using sum of roots, we have
$$\begin{aligned} 4+4+r &= -a \\ r &= 6-8 \\ &= -2 \end{aligned}$$

(c) (i)
$$x^2-x-6 = a\left(x^2-3\right)+(bx+c)(x+1)$$

Let $x=-1$,
$$\begin{aligned} (-1)^2-(-1)-6 &= a\left((-1)^2-3\right) \\ -4 &= a(-2) \\ a &= 2. \end{aligned}$$

By equating coefficients of x^2, we have
$$\begin{aligned} a+b &= 1 \\ b &= 1-a \\ &= 1-2 \\ &= -1. \end{aligned}$$

By equating coefficients of x, we have
$$\begin{aligned} b+c &= -1 \\ c &= -1-b \\ &= -1-(-1) \\ &= 0. \end{aligned}$$

Hence, $$\int \frac{x^2-x-6}{(x+1)(x^2-3)}dx = \int \frac{2}{x+1}+\frac{-x}{x^2-3}dx$$
$$= 2\log_e|x+1|-\frac{1}{2}\log_e|x^2-3|+c$$

(d) (i) $w = u \cdot i$
$= (5 + 2i)\, i$
$= -2 + 5i.$

(ii) $v = u + w$
$= 5 + 2i + -2 + 5i$
$= 3 + 7i.$

(iii) $\arg\left(\dfrac{w}{v}\right) = \arg w - \arg v.$

From the diagram, $\arg w - \arg v$ corresponds with the $\angle COB$, which is $45°$.

(e) By joining the interval CB, we form the angle CBA. Suppose $\angle CBA = \theta$. Then $\theta = \angle D$ as equal chords subtends equal angles at the circumference. From Pythagoras, we have

$$\begin{aligned}
\sin\theta &= \frac{AC}{AB} \\
&= \frac{d}{2r} \\
\sin D &= \frac{d}{2r} \\
d &= 2r\sin D.
\end{aligned}$$

Question 12

(a)

$$V = \pi \int_{-1}^{1} \text{Cross-Section } \Delta y$$
$$V = \pi \int_{-1}^{1} \frac{\sqrt{3}}{4} x^2 dy$$

Since $x = 1 - y^2$, then $x^2 = 1 - 2y^2 + y^4$.

$$V = \int_{-1}^{1} \frac{\sqrt{3}}{4}(1 - 2y^2 + y^4)dy$$
$$V = \frac{\sqrt{3}}{4}\left[y - \frac{2y^3}{3} + \frac{y^5}{5}\right]_{-1}^{1}$$
$$V = \frac{4\sqrt{3}}{15} u^3$$

(b) (i)

$$\begin{aligned} x^2 + xy + y^2 &= 3 \\ 2x + y + x\frac{dy}{dx} + 2y\frac{dy}{dx} &= 0 \\ x\frac{dy}{dx} + 2y\frac{dy}{dx} &= -2x - y \\ \frac{dy}{dx}(x + 2y) &= -(2x + y) \\ \frac{dy}{dx} &= -\frac{2x + y}{x + 2y} \end{aligned}$$

(ii) Let $\frac{dy}{dx} = 0$,

$$\begin{aligned} -\frac{2x + y}{x + 2y} &= 0 \\ 2x + y &= 0 \\ y &= -2x \end{aligned}$$

Substituting $y = 2x$ into $\frac{dy}{dx}$, we have

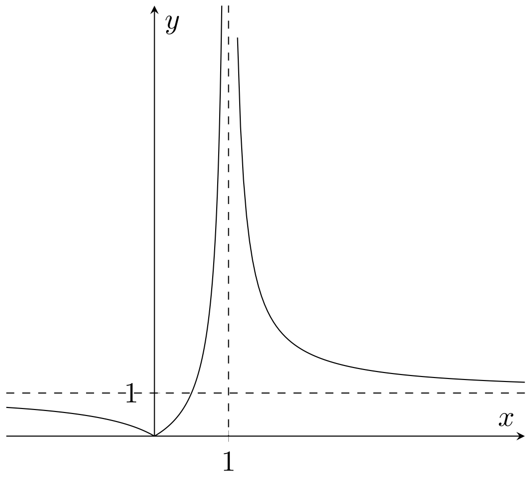
$$\begin{aligned} x^2 + x(-2x) + (-2x)^2 &= 3 \\ x^2 - 2x^2 + 4x^2 &= 3 \\ 3x^2 &= 3 \\ x^2 &= 1 \\ x &= \pm 1 \end{aligned}$$

Hence, we have the following stationary points at $(1, -2)$ and $(-1, 2)$.

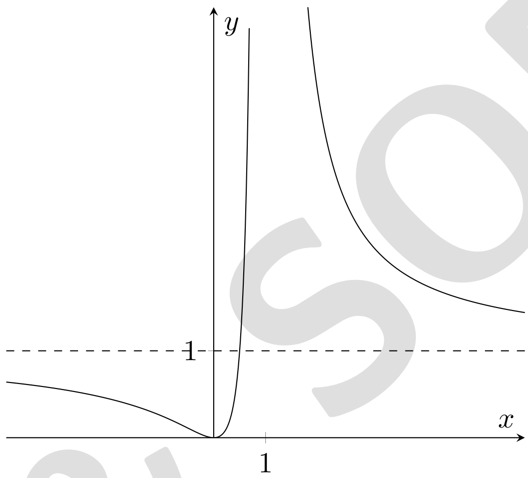
(c)

$$\begin{aligned} \int \frac{x^2 + 2x + 5 - 5}{x^2 + 2x + 5}\, dx &= \int 1 - \frac{5}{4 + (x + 1)^2}\, dx \\ &= x - \frac{5}{2}\tan^{-1}\left(\frac{x + 1}{2}\right) + C \end{aligned}$$

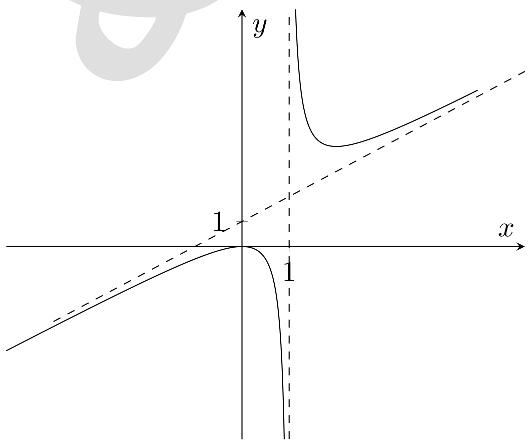
(d) (i)



(ii)

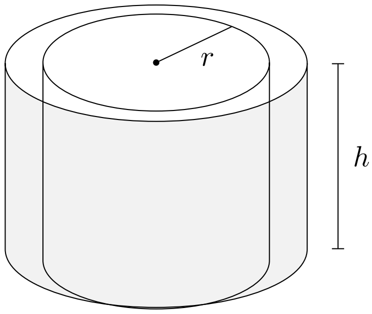


(iii)



Question 13

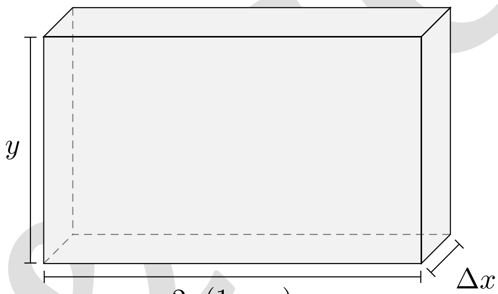
(a) (i) This question requires us to use the method of cylinders.
Using this technique, strips must be parallel to the axis rotation (here it is the x-axis) and then we cut it out. This is what the cylinder looks like when take the strip and rotate it around the y-axis.



Note: Volume of a cylinder: $V = 2\pi rh$

By 'flattening' the cylinder, we obtain a rectangular prism, where the circumference of the circle in the cylinder becomes the width of the rectangular prism.

$$V = \text{Cross-section} \times \text{height}$$



$$CA = 2\pi(1-x) \times (2y)$$
$$h = \Delta x$$

$$V = \int_0^1 2\pi(1-x)2y\,dx$$

Since $y^2 = x(1-x)^2$ then $y = \sqrt{x(1-x)^2}$.

$$\begin{aligned}
V &= 2\pi \int_0^1 (1-x)\ \times 2\sqrt{x(1-x)^2}\, dx \\
&= 4\pi \int_0^1 (1-x)^2 \sqrt{x}\, dx \\
&= 4\pi \int_0^1 (1-2x+x^2)\sqrt{x}\, dx \\
&= 4\pi \int_0^1 \sqrt{x} - 2x\sqrt{x} + x^2\sqrt{x}\, dx \\
&= 4\pi \int_0^1 x^{\frac{1}{2}} - 2x^{\frac{3}{2}} + x^{\frac{5}{2}}\, dx \\
&= 4\pi \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{4}{5}x^{\frac{5}{2}} + \frac{2}{7}x^{\frac{7}{2}}\right]_0^1 \\
&= 4\pi \left(\frac{2}{3} - \frac{4}{5} + \frac{2}{7}\right) \\
&= \frac{64}{105}\pi\, u^3
\end{aligned}$$

(b) (i)
$$\begin{aligned}
z &= 1 - \cos 2\theta + i\sin 2\theta \\
&= 2\sin^2\theta + i2\sin\theta\cos\theta \\
&= 2\sin\theta\,(\sin\theta + i\cos\theta) \\
|z| &= |2\sin\theta\,(\sin\theta + i\cos\theta)| \\
&= 2\sin\theta\,|\sin\theta + i\cos\theta| \\
&= 2\sin\theta\left(\sin^2\theta + \cos^2\theta\right) \\
&= 2\sin\theta.
\end{aligned}$$

(ii)
$$\begin{aligned}
\arg z &= \arg\left(2\sin\theta\,(\sin\theta + i\cos\theta)\right) \\
&= \arg(2\sin\theta) + \arg(\sin\theta + i\cos\theta) \\
&= 0 + \arg\left(\cos\left(\frac{\pi}{2} - \theta\right) + i\sin\left(\frac{\pi}{2} - \theta\right)\right) \\
&= \frac{\pi}{2} - \theta.
\end{aligned}$$

(c) (i) We have $F_x = t\sin\theta = mrw^2$, $F_y = T\cos\theta + N - mg = 0$, $N = mg - T\cos\theta$. To remain in contact, we need $N \geq 0$, and hence,

$$mg - T\cos\theta \geq 0.$$

From our equation of F_x, we have

$$T = \frac{mrw^2}{\sin\theta}.$$

$$\therefore mg - \frac{mrw^2\cos\theta}{\sin\theta} \geq 0.$$

Now we need to eliminate $\sin\theta$. From our diagram, $\sin\theta = \frac{r}{l}$, hence

$$mg - \frac{mrw^2\cos\theta}{\left(\frac{r}{l}\right)} \implies mg - mlw^2\cos\theta \geq 0 \implies w^2 \leq \frac{g}{l\cos\theta}.$$

(d) (i) Since PS is a distance between two points, then

$$\begin{aligned} PS &= \sqrt{(cp-0)^2 + \left(\frac{c}{p} - \frac{cp+cq}{pq}\right)^2} \\ &= \sqrt{c^2p^2 + \left(\frac{cp-cp-cq}{pq}\right)^2} \\ &= \sqrt{c^2p^2 + \frac{c^2}{q^2}} \end{aligned}$$

Similarly, for QR,

$$\begin{aligned} QR &= \sqrt{(c(p+q)-cq)^2 + \left(0 - \frac{c}{q}\right)^2} \\ &= \sqrt{(cp+cq-cq)^2 + \frac{c^2}{q^2}} \\ &= \sqrt{c^2p^2 + \frac{c^2}{q^2}} \\ &= PS. \end{aligned}$$

(ii) Let A and B have coordinates (x_A, y_A) and (x_B, y_B) respectively.
Since M is the midpoint of AB,
$M = \left(\dfrac{x_A + x_B}{2}, \dfrac{y_A + y_B}{2}\right)$
Solving $xy = c^2$ and $y = tx - at^2$ simultaneously,
$tx^2 - at^2x - c^2 = 0$
Since the tangent at T intersects $xy = c^2$ at A and B, the above quadratic in terms of x has roots x_A and x_B.
Considering the sum of roots,

$$\begin{aligned} x_A + x_B &= \frac{-(-at^2)}{2} \\ &= at \end{aligned}$$

$\therefore x_M = \frac{x_A+x_B}{2} = \frac{at}{2}$ Sub $x_M = \frac{at}{2}$ into $y = tx - at^2$

$$\begin{aligned} y_M &= tx_M - at^2 \\ &= \frac{at^2}{2} - at^2 \end{aligned}$$

$y_M = \frac{-at^2}{2}$
Since $x_M = \frac{at}{2}, t = \frac{2x_M}{a}$
$\therefore y_M = -\frac{a}{2}(\frac{2x_M}{a})^2$
$y_M = \frac{-2x_M^2}{a}$
$2x_M^2 = -ay_M$
$\therefore$ M lies on the parabola $2x^2 = -ay$

Question 14

(a) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{d\theta}{2-\cos\theta}$

Suppose

$$\begin{aligned} t &= \tan\frac{\theta}{2} \\ dt &= \frac{1}{2}\sec^2\frac{\theta}{2}\,d\theta \\ &= \frac{1}{2}\left(1+\tan^2\frac{\theta}{2}\right)\,d\theta \\ &= \frac{1}{2}(1+t^2)\,d\theta \\ d\theta &= \frac{2}{1+t^2}\,dt \end{aligned}$$

$$\begin{aligned} \text{We know that } \tan\theta &= \frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}} \\ &= \frac{2t}{1-t^2}. \end{aligned}$$

Hence $\cos\theta = \frac{1-t^2}{1+t^2}$.

Now changing the borders,

$$\begin{aligned} \theta = 0 &\implies t = 0 \\ \theta = \frac{\pi}{2} &\implies t = 1. \end{aligned}$$

$$\begin{aligned} \therefore \int_0^{\frac{\pi}{2}} \frac{d\theta}{2-\cos\theta} &= \int_0^1 \frac{\frac{2}{1+t^2}\,dt}{2-\frac{1-t^2}{1+t^2}} \\ &= \int_0^1 \frac{\frac{2}{1+t^2}\,dt}{\frac{2+2t^2-1+t^2}{1+t^2}} \\ &= \int_0^1 \frac{2\,dt}{1+3t^2} \\ &= 2\int_0^1 \frac{dt}{1+3t^2} \\ &= 2\frac{1}{\sqrt{3}}\left[\tan^{-1}(\sqrt{3}x)\right]_0^1 \\ &= \frac{2}{\sqrt{3}}(\tan^{-1}\sqrt{3}-\tan^{-1}0) \\ &= \frac{2}{\sqrt{3}}\left(\frac{\pi}{3}\right) \\ &= \frac{2\pi}{3\sqrt{3}} \end{aligned}$$

(b) Let the displacement of the particle be x. We will consider downwards as the positive direction.

$$\begin{aligned}
\therefore \ddot{x} &= g - kv^2 \\
\frac{v\,dv}{dh} &= g - kv^2 \\
dh &= \frac{v\,dv}{g - kv^2} \\
-2k\,dh &= \frac{-2kv\,dv}{g - kv^2} \\
\int -2k\,dh &= \int \frac{-2kv\,dv}{g - kv^2} \\
-2kh &= \ln(g - kv^2) + C
\end{aligned}$$

Initially, the particle is at rest, so when $h = 0$, $v = 0$. Substituting these conditions in, we get

$$\begin{aligned}
-2k(0) &= \ln(g - k(0)^2) + C \\
C &= -\ln(g).
\end{aligned}$$

Substituting C back into our equation, we get

$$\begin{aligned}
-2kh &= \ln(g - kv^2) + C \\
&= \ln(g - kv^2) - \ln(g) \\
&= \ln\left(\frac{g - kv^2}{g}\right)
\end{aligned}$$

$$\begin{aligned}
\ln\left(\frac{g - kv^2}{g}\right) &= -2kh \\
\frac{g - kv^2}{g} &= e^{-2kh} \\
1 - \frac{kv^2}{g} &= e^{-2kh} \\
\frac{kv^2}{g} &= 1 - e^{-2kh} \\
v^2 &= \frac{g}{k}\left(1 - e^{-2kh}\right) \\
v &= \pm\sqrt{\frac{g}{k}\left(1 - e^{-2kh}\right)}.
\end{aligned}$$

As the particle is falling downwards, then $v \geq 0$.

$$\therefore v = \sqrt{\frac{g}{k}(1 - e^{-2kh})}$$

as required.

(c) (i)

$$I_n = \int_{-3}^{0} x^n \sqrt{x+3}\,dx$$

Using integration by parts,

$$\begin{aligned}
u &= x^n & dv &= (x+3)^{\frac{1}{2}}\,dx \\
du &= nx^{n-1}\,dx & v &= \tfrac{2}{3}(x+3)^{\frac{3}{2}}.
\end{aligned}$$

$$\begin{aligned}
\therefore I_n &= \left[\frac{2}{3}x^n(x+3)^{\frac{3}{2}}\right]_{-3}^{0} - \frac{2}{3}n\int_{-3}^{0} x^{n-1}(x+3)^{\frac{3}{2}}\,dx \\
&= \left[\frac{2}{3}(0)^n(3)^{\frac{3}{2}} - \frac{2}{3}(-3)^n(0)^{\frac{3}{2}}\right] - \frac{2}{3}n\int_{-3}^{0} x^{n-1}(x+3)^{\frac{3}{2}}\,dx \\
&= -\frac{2}{3}n\int_{-3}^{0} x^{n-1}(x+3)^{\frac{3}{2}}\,dx \\
&= -\frac{2}{3}n\int_{-3}^{0} x^{n-1}(x+3)\sqrt{x+3}\,dx \\
&= -\frac{2}{3}n\left(\int_{-3}^{0} x^n\sqrt{x+3}\,dx + 3\int_{-3}^{0} x^{n-1}\sqrt{x+3}\,dx\right) \\
&= -\frac{2}{3}n(I_n + I_{n-1})
\end{aligned}$$

$$\begin{aligned}
I_n\left(1+\frac{2}{3}n\right) &= -2nI_{n-1} \\
I_n &= \frac{-2nI_{n-1}}{1+\frac{2}{3}n} \\
&= \frac{-6nI_{n-1}}{3+2n}
\end{aligned}$$

(ii) We know that

$$I_n = \frac{-6nI_{n-1}}{3+2n}.$$

Substituting $n = 2$, we get

$$I_2 = \frac{-6(2)I_1}{3+2(2)} = -\frac{12}{7}I_1.$$

Substituting $n = 1$, we get

$$I_1 = \frac{-6(1)I_0}{3+2(1)} = -\frac{6}{5}I_0.$$

Now we will find I_0 by substituting $n = 0$ into

$$I_n = \int_{-3}^{0} x^n\sqrt{x+3}\,dx,$$

which gives us

$$\begin{aligned}
I_0 &= \int_{-3}^{0} \sqrt{x+3}\,dx \\
&= \frac{2}{3}\left[(x+3)^{\frac{3}{2}}\right]_{-3}^{0} \\
&= \frac{2}{3}\cdot 3^{\frac{3}{2}} \\
&= \frac{2}{3}\cdot 3\cdot\sqrt{3} \\
&= 2\sqrt{3}
\end{aligned}$$

$$\begin{aligned}
\therefore I_1 &= -\frac{6}{5}(2\sqrt{3}) \\
&= -\frac{12}{5}\sqrt{3},
\end{aligned}$$

and hence,

$$\begin{aligned}
I_2 &= -\frac{12}{7}\cdot -\frac{12}{5}\sqrt{3} \\
&= \frac{144}{35}\sqrt{3}.
\end{aligned}$$

(d) (i) $\Pr(A \text{ wins } n \text{ games}) = \left(\frac{1}{3}\right)^n$.

(ii) $$\begin{aligned}\Pr(\text{C never wins}) &= \Pr(\text{A and B wins } n \text{ games})\\ &= \left(\frac{2}{3}\right)^n\end{aligned}$$

$$\begin{aligned}\Pr(A \text{ or } B \text{ wins all } n \text{ games}) &= \left(\frac{1}{3}\right)^n + \left(\frac{1}{3}\right)^n\\ &= 2\left(\frac{1}{3}\right)^n\end{aligned}$$

$$\Pr(C \text{ never wins but } A \text{ and } B \text{ wins at least one}) = \left(\frac{2}{3}\right)^n - 2\left(\frac{1}{3}\right)^n$$

(iii) $$\begin{aligned}\Pr(\text{Each player wins atleast one}) &= 1 - \Pr(\text{One players wins all } n \text{ games}) - \Pr(\text{Two player wins all } n \text{ game}\\ &= 1 - 3\cdot\left(\frac{1}{3}\right)^n - 3\left(\left(\frac{2}{3}\right)^n - 2\left(\frac{1}{3}\right)^n\right)\\ &= 1 - 3\left(\frac{2}{3}\right)^n + 3\left(\frac{1}{3}\right)^n\\ &= 1 - \frac{2^n}{3^{n-1}} + \frac{1}{3^{n-1}}\\ &= \frac{3^{n-1} - 2^n + 1}{3^{n-1}}\end{aligned}$$

Question 15

(a) (i) As $\angle AOB = \frac{\pi}{2}$, then the coordinates of B is given by

$$B(a\cos(\theta + \frac{\pi}{2}), b\sin(\theta + \frac{\pi}{2})) \Longrightarrow B(-a\sin\theta, a\cos\theta).$$

The x-coordinate of Q is the same as the x-coordinate of B (as Q is vertically above B). Now to find the y-coordinate of Q, we sub $x = -a\sin\theta$ into the equation of the ellipse, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. We get

$$\frac{a^2\sin^2\theta}{a^2} + \frac{y^2}{b^2} = 1 \Longrightarrow \frac{y^2}{b^2} = 1 - \sin^2\theta = \cos^2\theta \Longrightarrow y = \pm b\cos\theta.$$

As $y > 0$ (from the diagram), the y-coordinate of Q is $y = b\cos\theta$, and hence,

$$B(-a\sin\theta, b\cos\theta)$$

as required.

(ii) Let the angle between OP and the positive x-axis be α. Then

$$\tan\alpha = \frac{b\sin\theta}{a\cos\theta} = \frac{b}{a}\cdot\tan\theta \Longrightarrow \alpha = \tan^{-1}\left(\frac{b}{a}\cdot\tan\theta\right).$$

Let the angle between OQ and the negative x-axis be β. Then

$$\tan\beta = \frac{b\cos\theta}{a\sin\theta} = \frac{b}{a}\cdot\frac{1}{\tan\theta} \Longrightarrow \beta = \tan^{-1}\left(\frac{b}{a}\cdot\frac{1}{\tan\theta}\right)$$

Let the angle between OQ' and the positive x-axis be γ. Note that $\beta = \gamma$ (vertically opposite angles are equal). Now we want to find $\angle POQ'$. From the diagram,

$$\begin{aligned}
\angle POQ' &= \alpha + \gamma \\
&= \alpha + \beta \qquad (\text{as } \beta = \gamma) \\
&= \tan^{-1}\left(\frac{b}{a}\cdot\tan\theta\right) + \tan^{-1}\left(\frac{b}{a}\cdot\frac{1}{\tan\theta}\right) \\
\tan(\angle POQ') &= \tan\left(\tan^{-1}\left(\frac{b}{a}\cdot\tan\theta\right) + \tan^{-1}\left(\frac{b}{a}\cdot\frac{1}{\tan\theta}\right)\right) \\
&= \frac{\tan(\tan^{-1}(\frac{b}{a}\cdot\tan\theta)) + \tan(\tan^{-1}(\frac{b}{a}\cdot\frac{1}{\tan\theta}))}{1 - \tan(\tan^{-1}(\frac{b}{a}\cdot\tan\theta))\cdot\tan(\tan^{-1}(\frac{b}{a}\cdot\frac{1}{\tan\theta}))} \\
&= \frac{\frac{b}{a}\cdot\tan\theta + \frac{b}{a}\cdot\frac{1}{\tan\theta}}{1 - \frac{b}{a}\cdot\tan\theta + \frac{b}{a}\cdot\frac{1}{\tan\theta}} \\
&= \frac{\frac{b}{a}(\tan\theta + \frac{1}{\tan\theta})}{1 - \frac{b^2}{a^2}} \\
&= \frac{ab(\tan\theta + \frac{1}{\tan\theta})}{a^2 - b^2} \\
\angle POQ' &= \tan^{-1}\left(\frac{ab(\tan\theta + \frac{1}{\tan\theta})}{a^2 - b^2}\right)
\end{aligned}$$

Now to minimise $\angle POQ'$, we must minimise $\tan\theta + \frac{1}{\tan\theta}$. To minimise this, we will consider the AM-GM inequality

$$\frac{x+y}{2} \geq \sqrt{xy}.$$

Substituting $x = \tan\theta$ and $y = \frac{1}{\tan\theta}$, we get

$$\frac{\tan\theta + \frac{1}{\tan\theta}}{2} \geq \sqrt{\tan\theta\cdot\frac{1}{\tan\theta}} = 1 \Longrightarrow \tan\theta + \frac{1}{\tan\theta} \geq 2.$$

Therefore the minimal value of $\tan\theta + \frac{1}{\tan\theta}$ is 2. Hence the minimum size of $\angle POQ'$ is

$$\tan^{-1}\left(\frac{2ab}{a^2-b^2}\right).$$

(b) (i) Using de Moivre's theorem,

$$(\cos\theta + i\sin\theta)^8 = \cos 8\theta + i\sin 8\theta.$$

Using the binomial theorem,

$$(\cos\theta + i\sin\theta)^8 = \sum_{k=0}^{8}\binom{8}{k}(\cos\theta)^{8-k}(i\sin\theta)^k.$$

Equating these,

$$\cos 8\theta + i\sin 8\theta = \sum_{k=0}^{8} i^k\binom{8}{k}\cos^{8-k}\theta\,\sin^k\theta.$$

Since $i^2=-1$, only even powers of i are real, meaning odd powers of i are not real. Then equating imaginary parts above and using $i=i, i^3=-i, i^5=i, i^7=-i$,

$$\sin 8\theta = \binom{8}{1}\cos^7\theta\,\sin\theta - \binom{8}{3}\cos^5\theta\,\sin^3\theta + \binom{8}{5}\cos^3\theta\,\sin^5\theta - \binom{8}{7}\cos\theta\,\sin^7\theta.$$

(ii) Note $\binom{8}{1}=\binom{8}{7}=8$ and $\binom{8}{3}=\binom{8}{5}=56$. Then, dividing the result in (i) by $\sin 2\theta$,

$$\begin{aligned}
\frac{\sin 8\theta}{\sin 2\theta} &= \frac{8\cos^7\theta\,\sin\theta - 56\cos^5\theta\,\sin^3\theta + 56\cos^3\theta\,\sin^5\theta - 8\cos\theta\,\sin^7\theta}{\sin 2\theta}\\
&= \frac{8\cos^7\theta\,\sin\theta - 56\cos^5\theta\,\sin^3\theta + 56\cos^3\theta\,\sin^5\theta - 8\cos\theta\,\sin^7\theta}{2\sin\theta\cos\theta} \quad \text{(double angle)}\\
&= 4\cos^6\theta - 28\cos^4\theta\,\sin^2\theta + 28\cos^2\theta\,\sin^4\theta - \sin^6\theta\\
&= 4\left((1-\sin^2\theta)^3 - 7(1-\sin^2\theta)^2\sin^2\theta + 7(1-\sin^2\theta)\sin^4\theta - \sin^6\theta\right) \quad (\cos^2\theta+\sin^2\theta=1)\\
&= 4((\sin^6\theta)(-1-7-7-1) + (\sin^4\theta)(3-7(-2)+7) + (\sin^2\theta)(-3-7)+1)\\
&= 4(1-10\sin^2\theta + 24\sin^4\theta - 16\sin^6\theta)
\end{aligned}$$

(c) (i)
$$\begin{aligned}
x^n - 1 - n(x-1) &= (x-1)(1+x+x^2+\cdots+x^{n-1}) - n(x-1)\\
&= (x-1)((1+x+x^2+\cdots+x^{n-1})-n)
\end{aligned}$$

(ii) It is sufficient to prove that $(x-1)(1+x+x^2+\cdots+x^{n-1}-n)\geq 0$.

- Case 1: $x<1$.
  Then $x-1<0$ and each of the terms x^i will be less than 1. Then $1+x+x^2+\cdots+x^{n-1} < \underbrace{1+1+\cdots+1}_{n\text{ times}} = n$, so

  $$1+x+x^2+\cdots+x^{n-1}-n<0$$

  As both the terms $(x-1)$ and $(1+x+x^2+\cdots+x^{n-1}-n$ are negative, their product $(x-1)(1+x+x^2+\cdots+x^{n-1}-n)$ is positive.

- Case 2: $x\geq 1$.
  Then $x-1>0$.
  Also, for $x>1$, then $x^i>1$ for $1<i<n$. Then the sum $1+x+x^2+\cdots+x^{n-1} > \underbrace{1+1+\cdots+1}_{n\text{ times}} = n$, so

  $$1+x+x^2+\cdots+x^{n-1}-n>0$$

  As both the terms $(x-1)$ and $(1+x+x^2+\cdots+x^{n-1}-n$ are positive, their product $(x-1)(1+x+x^2+\cdots+x^{n-1}-n)$ is positive.

In both cases, we have proved that $(x-1)(1+x+x^2+\cdots+x^{n-1}-n)\geq 0$. Hence

$$\begin{aligned} x^n-1-n(x-1) &\geq 0 \\ x^n &\geq 1+n(x-1) \end{aligned}$$

(iii) Method 1: Let $x=ab^{-1}$.

$$\begin{aligned} a^nb^{-n} &\geq 1+n\left(\frac{a}{b}-1\right) \\ a^nb^{1-n} &\geq b+n(a-b) \\ &= b+na-nb \\ &= na+(1-n)b \end{aligned}$$

Question 16

(a)

$$x^{(3^n)} - 1 = (x-1)(x^2+x+1)(x^6+x^3+1)\cdots(x^{(2\times 3^{n-1})} + x^{(3^{n-1})} + 1)$$

First, we will prove that the statement holds true for $n = 1$. Letting $n = 1$, we get

$$\begin{aligned} \text{LHS} &= x^{(3^1)} - 1 \\ &= x^3 - 1 \\ \text{RHS} &= (x-1)(x^2+x+1) \\ &= x^3 - 1 \\ &= \text{LHS}. \end{aligned}$$

As LHS = RHS, the statement holds true for $n = 1$.

We will assume that the statement holds true for $n = k$, that is,

$$x^{(3^k)} - 1 = (x-1)(x^2+x+1)(x^6+x^3+1)\cdots\left(x^{(2\times 3^{k-1})} + x^{(3^{k-1})} + 1\right).$$

Now we will prove that the statement holds true for $n = k+1$, that is,

$$x^{(3^{k+1})} - 1 = (x-1)(x^2+x+1)(x^6+x^3+1)\cdots(x^{(2\times 3^{k-1})} + x^{(3^{k-1})} + 1)(x^{(2\times 3^k)} + x^{(3^k)} + 1).$$

$$\begin{aligned} \text{LHS} &= x^{(3^{k+1})} - 1 \\ &= x^{(3^k\times 3)} - 1 \\ &= (x^{(3^k)})^3 - 1 \\ &= \underbrace{(x^{(3^k)} - 1)}_{\text{assumption}}((x^{(3^k)})^2 + x^{(3^k)} + 1) \\ &= (x-1)(x^2+x+1)(x^6+x^3+1)\cdots(x^{(2\times 3^{k-1})} + x^{(3^{k-1})} + 1)((x^{(3^k)})^2 + x^{(3^k)} + 1) \\ &= (x-1)(x^2+x+1)(x^6+x^3+1)\cdots(x^{(2\times 3^{k-1})} + x^{(3^{k-1})} + 1)(x^{(2\times 3^k)} + x^{(3^k)} + 1) \\ &= \text{RHS} \end{aligned}$$

As our statement holds true for $n = k+1$, we have completed the proof by induction.

(b) (i) As BA is parallel to HI, then ΔABC is similar to ΔIHC. Hence, we can say that

$$\frac{BC}{HC} = \frac{BA}{HI} = \sqrt{2} \qquad \text{(corresponding sides of similar triangles are equal).}$$

Similarly, as FG is parallel to CA, then

$$\frac{BC}{BF} = \frac{CA}{FG} = \sqrt{2} \qquad \text{(corresponding sides of similar triangles are equal).}$$

$$\therefore \frac{BC}{HC} = \frac{BC}{BF} \Longrightarrow HC = BF.$$

As $HC = HF + FC$ and $BF = BH + HF$, then

$$HF + FC = BH + HF \Longrightarrow BH = FC.$$

Since DE is parallel to BC, FG is parallel to CA and HI is parallel to BA, then $BDYH$ and $CEZF$ are parallelograms. In $BDYH$, $BH = DY$ and in $CEZF$, $FC = ZE$. Thus,

$$BH = FC \Longrightarrow DY = ZE$$

as required.

(ii) Let $BH = FC = DY = ZE = x$. Then

$$BC = BH + HF + FC = 2x + HF,$$

$$HC = HF + FC = x + HF,$$

$$DE = DY + YZ + ZE = 2x + YZ.$$

From (i),

$$\begin{aligned}\frac{BC}{HC} &= \sqrt{2}\\ \frac{2x+HF}{x+HF} &= \sqrt{2}\\ 2x + HF &= \sqrt{2}x + \sqrt{2}SF\\ HF &= \frac{(2-\sqrt{2})x}{\sqrt{2}-1}\\ &= \sqrt{2}x.\end{aligned}$$

$$\therefore BC = 2x + \sqrt{2}x = (2+\sqrt{2})x.$$

Also, we were given that

$$\frac{BC}{DE} = \sqrt{2}.$$

$$\begin{aligned}\therefore \frac{2x+\sqrt{2}x}{2x+YZ} &= \sqrt{2}\\ 2x + \sqrt{2}x &= 2\sqrt{2}x + \sqrt{2}YZ\\ YZ &= \frac{(2-\sqrt{2})x}{\sqrt{2}}\\ &= (\sqrt{2}-1)x\end{aligned}$$

$$\therefore \frac{YZ}{BC} = \frac{(\sqrt{2}-1)x}{(2+\sqrt{2})x} = \frac{\sqrt{2}-1}{2+\sqrt{2}} = \frac{3\sqrt{2}-4}{2}$$

(c) $p(x) = x^3 + px + q$, p, q real, $q \neq 0$, zeroes α, β, γ.

(i) Using relationships between roots and coefficients, we know that

$$\begin{aligned}\alpha + \beta + \gamma &= 0\\ \alpha\beta + \beta\gamma + \alpha\gamma &= p\\ \alpha\beta\gamma &= -q.\end{aligned}$$

Then consider

$$\begin{aligned}(\beta-\gamma)^2 &= \beta^2 + \gamma^2 - 2\beta\gamma\\ &= (\alpha^2+\beta^2+\gamma^2) - \alpha^2 - 2\cdot\frac{\alpha\beta\gamma}{\alpha}\\ &= (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\gamma\alpha)^2 - \alpha^2 - 2\cdot\frac{-q}{\alpha}\\ &= -2p - \alpha^2 + \frac{2q}{\alpha}.\end{aligned}$$

We now need to eliminate p from the expression but require only α terms in the expression, so we don't use sum of roots 2 at a time again. Instead, note $p(\alpha) = 0$ as α is a zero.

$$\begin{aligned}\alpha^3 + p\alpha + q &= 0\\ p &= -\frac{q}{\alpha} - \alpha^2\end{aligned}$$

Thus

$$\begin{aligned}(\beta-\gamma)^2 &= -2\left(-\frac{q}{\alpha}-\alpha^2\right)-\alpha^2+\frac{2q}{\alpha}\\ &= \alpha^2+\frac{4q}{\alpha}.\end{aligned}$$

(ii) We can further show

$$\begin{aligned}(\beta-\gamma)^2 &= \alpha^2+\frac{4q}{\alpha}\\ &= \frac{\alpha^3+4q}{\alpha}\\ &= \frac{(-p\alpha-q)+4q}{\alpha} \text{ (as } p(\alpha)=0)\\ &= -p+\frac{3q}{\alpha}.\end{aligned}$$

A similar result applies to $(\alpha-\beta)^2$ and $(\gamma-\alpha)^2$ we know that the roots of the equation of interest can be rewritten as $-p+\dfrac{3q}{\alpha}, -p+\dfrac{3q}{\beta}, -p+\dfrac{3q}{\gamma}$.

Set $y=-p+\dfrac{3q}{x}$ so $x=\dfrac{3q}{y+p}$. Then using $p(x)=0$, we have

$$\left(\frac{3q}{y+p}\right)^2+p\left(\frac{3q}{y+p}\right)+q=0$$

$\times(y+p)^3 \div q: \quad 27q^2+3p(y+p)^2+(y+p)^3=0.$

The roots of this equation are $y=(\alpha-\beta)^2,(\beta-\gamma)^2,(\gamma-\alpha)^2$.
Taking product of roots,

$$\begin{aligned}(\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\alpha)^2 &= -\frac{\text{constant term}}{\text{leading coefficient}}\\ &= -\frac{27q^2+3p\cdot p^2+p^3}{1}\\ &= -(27q^2+4p^3).\end{aligned}$$

(iii) If $27q^2+4p^3<0$ then

$$\begin{aligned}(\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\alpha)^2 &= -(27q^2+4p^3)\\ &> 0.\end{aligned}$$

Firstly, note from this that

$$(\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\alpha)^2 \neq 0.$$

which means that $\alpha\neq\beta$, $\beta\neq\gamma$, and $\gamma\neq\alpha$. In other words, no 2 zeroes of $p(x)=0$ are equal so all 3 zeroes are distinct.
Next, because $p(x)$ has coefficients that are all real, any non-real roots must occur in conjugate pairs, so there are either 3 real distinct roots or 1 real root with 2 conjugate complex roots.
If we suppose we had non-real roots $\beta=a+ib$ and $\gamma=a-ib$ (a,b real) then

$$\begin{aligned}(\alpha-\beta)^2(\alpha-\gamma)^2(\beta-\gamma)^2 &= (\alpha-(a+ib))^2(\alpha-(a-ib))^2((a+ib)-(a-ib))^2\\ &= \left((\alpha-(a+ib))(\alpha-(a-ib))\right)^2(2ib)^2\\ &= -(\alpha^2-2\alpha+(a^2+b^2))^2(4b^2)\\ &< 0.\end{aligned}$$

The inequality arises because squares of real numbers are non-negative. However, the sign of the expression is given to be positive, so we cannot have any non-real roots. Therefore, $p(x)=0$ has 3 real distinct zeroes if $27q^2+4p^3<0$.