2018 HSC Mathematics Extension 1 Solutions

Multiple Choice

Multiple Choice Answer Key

Question	Answer
1	B
2	A
3	A
4	D
5	A
6	C
7	C
8	B
9	D
10	B

Explanation

1. $\alpha\beta\gamma = -\frac{(-10)}{2} = 5$ from product of roots, and
$\alpha + \beta + \gamma = -\frac{6}{2} = -3$.

Hence $\alpha\beta\gamma(\alpha + \beta + \gamma) = 5 \times -3 = -15$.

Hence B.

2. $y = 3x \implies m_1 = 3$
$y = 5x \implies m_2 = 5$.

$$\begin{aligned}\tan\theta &= \left|\frac{m_1 - m_2}{1 + m_2 m_2}\right| \\ &= \left|\frac{3-5}{1+3(5)}\right| \\ &= \left|\frac{-2}{16}\right| \\ &= \frac{1}{8}\end{aligned}$$

Hence A.

3. $$\begin{aligned}\lim_{x\to 0}\frac{\sin 3x\cos 3x}{12x} &= \lim_{x\to 0}\frac{2\sin 3x\cos 3x}{24x}\\ &= \lim_{x\to 0}\frac{1}{4}\times\frac{\sin 6x}{6x}\\ &= \frac{1}{4}\times 1\\ &= \frac{1}{4}\end{aligned}$$

Hence A.

4. The roots of the polynomial are at $x=-2, x=-1$ and $x=1$, suggesting factors of $(x+2)$ and $(x+1)$ and $(x-1)$. The 'bounce' at $x=1$ suggests an even power for $(x-1)$, so $(x-1)^2$ is a possible factor.

The polynomial at the moment takes the form $y=a(x+2)(x+1)(x-1)^2$.

Since the curve has a y-intercept of $(0,-6)$, we have

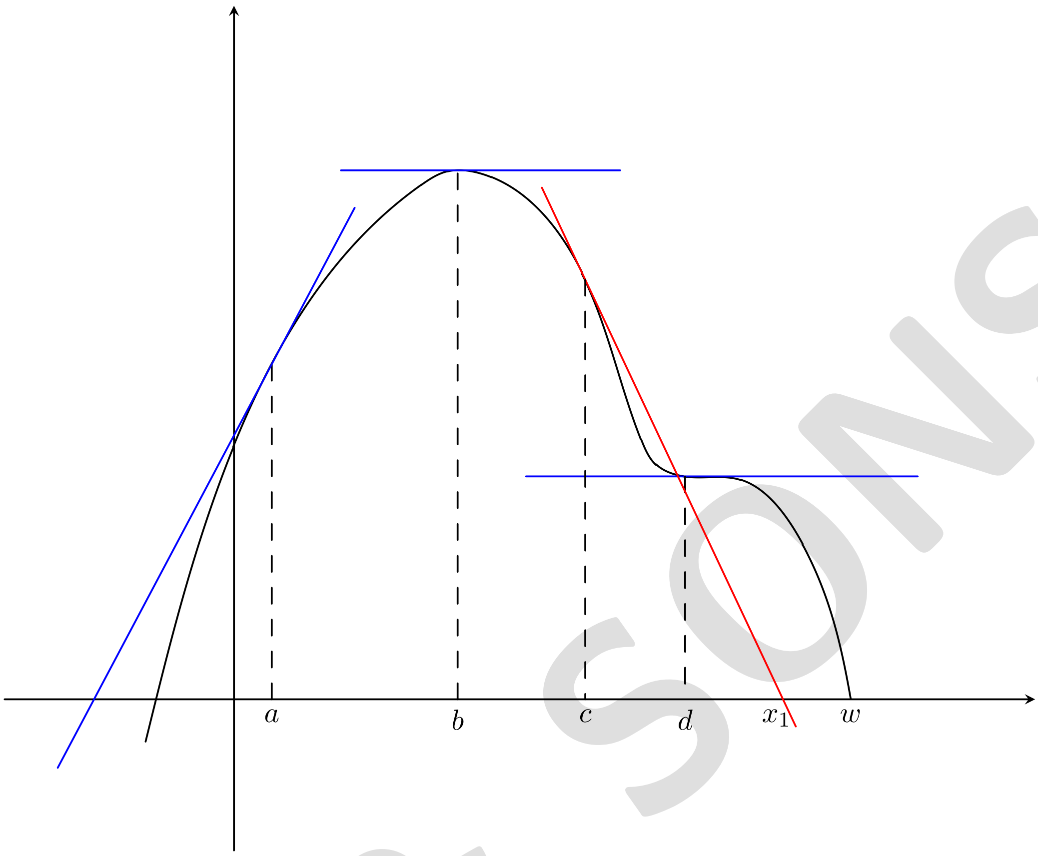
$$\begin{aligned}-6 &= a(0+2)(0+1)(0-1)^2\\ \implies -6 &= a\times 2\times 1\times 1\\ \implies a &= -3\end{aligned}$$

Therefore $a=-3, b=2, c=1, d=-1$.
Hence D.

5.
- The intercept of $P=3000$ when $t=0$ suggests that eliminates the option (C), which has an intercept of 4500.
- The horizontal asymptote of $P=1500$ as $t\to\infty$ implies (A) is the answer.

6. $x_1 = c$ will give the closest approximation to the solution $x = w$. The Newton-Raphson method of approximating roots works based on drawing a tangent line to the curve at $x_1 = g$ (where $x = g$ is an initial approximation), then using the x-intercept of the tangent line as an approximation of the root.

The diagram shows the tangent line drawn at $x_1 = c$ in red, with the other tangent lines drawn in blue:



- Option (a) results in a root to the left of a, even further from w.
- Option (b) results in a horizontal tangent line, giving no approximation at all.
- Option (c) produces a root close to w. (The red tangent)
- Option (d) produces a horizontal tangent line, giving no approximation at all.

Hence C.

7. Since, we are given $v(x)$ equation, to find the function for acceleration, we chase for $v\dfrac{dv}{dx}$.

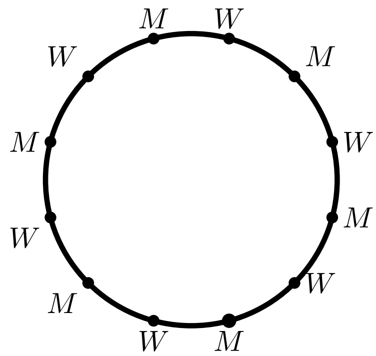
$$v = x^2 + 2$$
$$\frac{dv}{dx} = 2x$$
$$v\frac{dv}{dx} = 2xv$$
$$v\frac{dv}{dx} = 2x(x^2+2)$$
$$a = 2x(x^2+2)$$

At $x = 1$, $a = 2(1)(1+2)$

$$a = 6\,\text{ms}^{-2}$$

Hence C.

8. Fix the first person in a particular spot on the table. Without loss of generality, let this person be a male. Then the remaining 5 males can be permuted in 5! ways, and the 6 females can be permuted in 6! ways. Therefore, the total number of ways to seat men and women when alternating is 5! 6!. Hence B.



9. In general, for $\sin\theta = a$, then

$$\theta = n\pi + (-1)^n \sin^{-1} a.$$

Then

$$\begin{aligned}\sin 2x &= -\frac{1}{2}\\ 2x &= n\pi + (-1)^n \sin^{-1}\left(-\frac{1}{2}\right)\\ 2x &= n\pi + (-1)^n\left(-\frac{\pi}{6}\right)\\ x &= \frac{n\pi}{2} + (-1)^{n+1}\frac{\pi}{12}.\end{aligned}$$

Hence D.

10. Method 1.

Recall that the equation for a particle undergoing simple harmonic motion is $v^2 = n^2(a^2 - (x - x_0)^2)$.

$$\begin{aligned} v^2 &= -n^2(x^2 - 2kx) \\ &= -n^2((x-k)^2 - k^2) \qquad \text{(completing the square)} \\ &= n^2(k^2 - (x-k)^2) \end{aligned}$$

This means that amplitude a is k, and centre of motion is also $x_0 = k$.
The only option that agrees with this is B.
Hence B.

Method 2.

Note that, when $t = 0, x = k$.

$$\begin{aligned} v &= n\sqrt{2kx - x^2} \\ \frac{dx}{dt} &= n\sqrt{2kx - x^2} \\ \int \frac{dx}{\sqrt{k^2 - (x-k)^2}} &= \int n\, dt \\ \sin^{-1}\frac{x-k}{k} &= nt + c \\ \frac{x-k}{k} &= \sin(nt + c) \\ x - k &= k\sin(nt + c) \\ x &= k + k\sin(nt + c) \end{aligned}$$

Since $x = k$ at $t = 0$,

$$\begin{aligned} k &= k + k\sin(c) \\ \sin c &= 0 \\ c &= 0 \\ \therefore\ x &= k + k\sin(nt) \end{aligned}$$

Hence B.

Question 11

(a) (i) $P(x) = x^3 - 2x^2 - 5x + 6$
By the factor theorem,

$$\begin{aligned} P(1) &= 1^3 - 2\cdot 1^2 - 5\cdot 1 + 6 \\ &= 1 - 2 - 5 + 6 \\ &= 0. \end{aligned}$$

Since $P(1) = 0$, by the factor theorem $x - 1$ is a zero of $P(x)$.

(ii) By dividing the polynomial $P(x)$ by $x - 1$, we have

$$\begin{aligned} P(x) &= (x-1)\left(x^2 - x - 6\right) \\ &= (x-1)(x+2)(x-3). \end{aligned}$$

Hence, the other factors are $x = -2$ and $x = 3$.

(b)
$$\begin{aligned} \log_2 5 + \log_2 (x-2) &= 3 \\ \log_2(5(x-2)) &= 3 \\ 5(x-2) &= 2^3 \\ 5x - 10 &= 8 \\ 5x &= 18 \\ x &= \frac{18}{5}. \end{aligned}$$

Note: $x - 2 > 0$ so $x > 2$.
$\therefore x = \dfrac{18}{5}$.

(c)
$$\begin{aligned} \sqrt{3}\sin(x) + \cos(x) &\equiv R\sin(x+\alpha) \\ &\equiv R\sin x\cos\alpha + R\sin\alpha\cos x. \end{aligned}$$

By equating coefficients of $\sin x$ and $\cos x$,

$$R\cos\alpha = \sqrt{3}$$
$$R\sin\alpha = 1$$

By dividing the above two equations, we have

$$\begin{aligned} \tan\alpha &= \frac{1}{\sqrt{3}} \\ \alpha &= \frac{\pi}{6} \qquad (0 \le \alpha \le \frac{\pi}{2}). \end{aligned}$$

By considering the identity $R^2 = R^2\cos^2 x + R^2\sin^2 x$

$$\begin{aligned} R^2 &= (\sqrt{3})^2 + (1)^2 \\ R &= 2 \qquad (R > 0). \end{aligned}$$

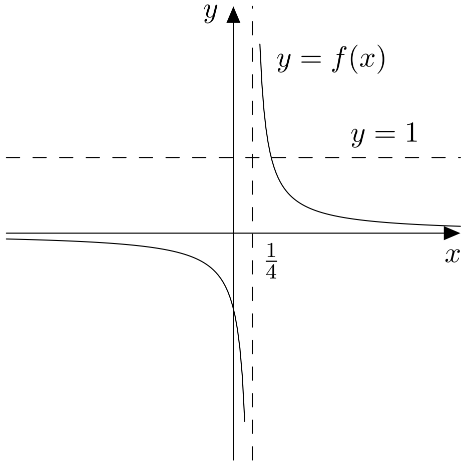
Hence, we have $\sqrt{3}\sin x + \cos x = 2\sin\left(x + \dfrac{\pi}{6}\right)$.

(d) Since external secants intersect in equal ratio, then

$$\begin{aligned} (2+x)\,x &= (5+3)\,3 \\ x^2 + 2x &= 24 \\ x^2 + 2x - 24 &= 0 \\ (x+6)(x-4) &= 0 \\ x &= 4 \qquad (x > 0). \end{aligned}$$

(e) (i) The domain of $f(x)$ is all real values of x, except $4x-1=0 \Longrightarrow x=\frac{1}{4}$. Hence all real x, where $x \neq \frac{1}{4}$.

(ii) **Method 1** (Geometry).
To find the values of x such that $f(x)<1$, we will firstly sketch $y=f(x)$.



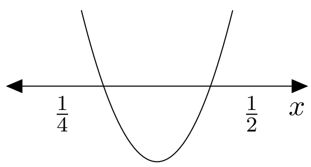
Now solving $f(x)=1$, we get

$$\frac{1}{4x-1}=1 \Longrightarrow 4x-1=1 \Longrightarrow x=\frac{1}{2}.$$

Hence, $x<\frac{1}{4}$ or $x>\frac{1}{2}$ for $f(x)<1$.

Method 2 (Algebra).

$$\begin{aligned}
f(x) &< 1 \\
\frac{1}{4x-1} &< 1 \\
(4x-1) &< (4x-1)^2 \\
(4x-1)^2-(4x-1) &> 0 \\
(4x-1)[(4x-1)-1] &> 0 \\
(4x-1)(4x-2) &> 0 \\
2(4x-1)(2x-1) &> 0
\end{aligned}$$



Hence, $x<\frac{1}{4}$ or $x>\frac{1}{2}$ for $f(x)<1$.

(f)

$$\int_{-3}^{0} \frac{x}{\sqrt{1-x}}\, dx$$

Now letting $u = 1 - x$, we get $du = -dx$. Changing the borders, we then have

$$x = -3 \Longrightarrow u = 4$$

$$x = 0 \Longrightarrow u = 1.$$

Hence, our integral becomes

$$\begin{aligned}\int_{-3}^{0} \frac{x}{\sqrt{1-x}}\, dx &= \int_{4}^{1} \frac{1-u}{\sqrt{u}} \cdot -du \\ &= \int_{1}^{4} \frac{1-u}{\sqrt{u}} \cdot du \\ &= \int_{1}^{4} u^{-\frac{1}{2}} - u^{\frac{1}{2}}\, du \\ &= \left[2u^{\frac{1}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right]_{1}^{4} \\ &= -\frac{8}{3}.\end{aligned}$$

Question 12

(a) $\displaystyle\int \cos^2(3x)\,dx = \frac{1}{2}\int \cos(6x) + 1\,dx$

$$= \frac{1}{2}\left(\frac{\sin(6x)}{6} + x\right) + C.$$

(b) (i) Considering the given triangle, we have

$$\sin\theta = \frac{h}{20} \implies h = 20\sin\theta \implies \frac{dh}{d\theta} = 20\cos\theta$$

as required.

(ii) Now we want to find $\frac{dh}{dt}$ when $h = 15$. We know that

$$\frac{dh}{dt} = \frac{dh}{d\theta}\cdot\frac{d\theta}{dt}.$$

It's known that $\frac{dh}{d\theta} = 20\cos\theta$ and $\frac{d\theta}{dt} = 1.5$. Hence,

$$\frac{dh}{dt} = 20\cos\theta \times 1.5 = 30\cos\theta.$$

We need $\frac{dh}{dt}$ to be in terms of h, so from our triangle we have that

$$\cos\theta = \frac{\sqrt{20^2 - h^2}}{20},$$

and thus

$$\frac{dh}{dt} = 30 \times \frac{\sqrt{20^2 - h^2}}{20}.$$

When $h = 15\,\text{m}$, then

$$\frac{dh}{dt}(h = 15) = 30 \times \frac{\sqrt{20^2 - 15^2}}{20} = 19.8\,\text{m/min} \quad (1\text{ d.p}).$$

Therefore, the speed of the top of the carriage rising when it is 15 metres higher than the horizontal diameter is 19.8 m/min.

(c) (i) $f(x) = \sin^{-1} x + \cos^{-1} x$

$$f'(x) = \left(\frac{1}{\sqrt{1-x^2}}\right) + \left(-\frac{1}{\sqrt{1-x^2}}\right)$$
$$= 0.$$

(ii) $$\frac{d}{dx}(\sin^{-1} x + \cos^{-1} x) = 0$$
$$d(\sin^{-1} x + \cos^{-1} x) = dx$$
$$\int d(\sin^{-1} x + \cos^{-1} x) = \int dx$$
$$\sin^{-1} x + \cos^{-1} x = C,$$

for some constant C. Now we will sub in a value for x to find C. Substituting $x = 0$, we get

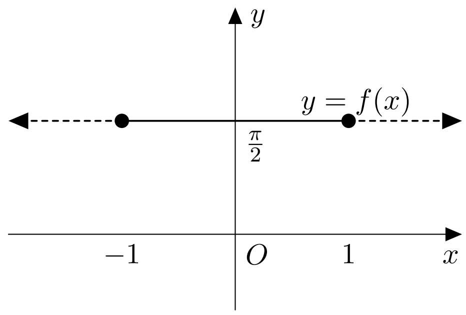
$$\sin^{-1} 0 + \cos^{-1} 0 = C \implies C = \frac{\pi}{2}.$$

Hence we have

$$\sin^{-1} x + \cos^{-1} x = C \implies \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$$

as required.

(iii) Since $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$, then $f(x) = \frac{\pi}{2}$. Hence, our sketch of $y = f(x)$ will be $y = \frac{\pi}{2}$.



Note, that the domain for inverse sine and cosine is $-1 \le x \le 1$. So when we sketch $y = \frac{\pi}{2}$ we must restrict our domain.

(d)
$$\begin{aligned}\Pr(\text{At least 10 people}) &= \Pr(\text{10 people}) + \Pr(\text{11 people}) + \Pr(\text{12 people}) \\ &= \binom{12}{10}(0.75)^{10}(0.25)^2 + \binom{12}{11}(0.75)^{11}(0.25)^1 + \binom{12}{12}(0.75)^{12}(0.25)^0.\end{aligned}$$

Hence, the expression for the probability that at least 10 people from the group complete the trek within 8 hours is given by

$$\binom{12}{10}(0.75)^{10}(0.25)^2 + \binom{12}{11}(0.75)^{11}(0.25)^1 + \binom{12}{12}(0.75)^{12}(0.25)^0.$$

In this question, note that the question requires an expression, meaning that we do not actually have to evaluate the above. For the sake of completeness,

$$\binom{12}{10}(0.75)^{10}(0.25)^2 + \binom{12}{11}(0.75)^{11}(0.25)^1 + \binom{12}{12}(0.75)^{12}(0.25)^0 = 0.39 \quad \text{(2 dp)}.$$

(e) (i)

$$x^2 = 4ay$$

First we will find the tangent at P. By differentiating and substituting in the coordinates of P, we have

$$\begin{aligned} y &= \frac{x^2}{4a} \\ \frac{dy}{dx} &= \frac{x}{2a} \\ \frac{dy}{dx}(x = 2ap) &= \frac{2ap}{2a} \\ &= p. \end{aligned}$$

Now the equation of the tangent at P is given by

$$\begin{aligned} y - ap^2 &= p(x - 2ap) \\ y &= px - 2ap^2 + ap^2 \\ &= px - ap^2. \end{aligned}$$

Now to find the coordinates of A, we let $y = 0$, giving us

$$\begin{aligned} px - ap^2 &= 0 \\ px &= ap^2 \\ x &= ap. \end{aligned}$$

$$\therefore A(ap, 0).$$

Now we calculate the gradient of AS,

$$\begin{aligned} m_{AS} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{0 - a}{ap - 0} \\ &= \frac{-a}{ap} \\ &= -\frac{1}{p}. \end{aligned}$$

The gradient of AP is the gradient of the tangent at P. Hence

$$m_{AP} = p.$$

Now since

$$m_{AS} \times m_{AP} = -\frac{1}{p} \times p = -1,$$

then $\angle PAS = 90°$.

(ii) Similarly, $\angle QBS = 90°$. Therefore, S, B, A and T are concyclic points as the interval ST subtends equal angles at two points A and B, on the same side.

(iii) Since ST is a chord of the circle through S, B, A and T that subtends a right angle at the circumference at the points A and B, ST is a diameter of the circle.

$$\begin{aligned} d_{ST} &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\ &= \sqrt{(0 - a(p+q))^2 + (a - apq)^2} \\ &= \sqrt{a^2p^2 + 2a^2pq + a^2q^2 + a^2q^2 - 2a^2pq + a^2p^2q^2} \\ &= a\sqrt{p^2 + q^2 + 1 + p^2q^2} \qquad (|a| = a \text{ as } a > 0) \\ &= a\sqrt{(p^2+1)(q^2+1)} \end{aligned}$$

Hence, the diameter of the circle through S, B, A and T has length

$$a\sqrt{(p^2+1)(q^2+1)}$$

as required.

Question 13

(a) Step 1: Prove true for $n = 1$

$$\begin{aligned} \text{LHS} &= 2(-3)^{1-1} \\ &= 2 \\ \text{RHS} &= \frac{1-(-3)^1}{2} \\ &= 2 \\ &= \text{LHS} \\ \therefore \text{True for } n &= 1 \end{aligned}$$

Step 2: Assume true for $n = k$

$$2 - 6 + 18 - 54 + \cdots + 2(-3)^{k-1} = \frac{1-(-3)^k}{2}$$

Step 3: Prove true for $n = k+1$

RTP: $2 - 6 + 18 - 54 + \cdots + 2(-3)^{k-1} + 2(-3)^k = \dfrac{1-(-3)^{k+1}}{2}$

$$\begin{aligned} \text{LHS} &= 2 - 6 + 18 - 54 + \cdots + 2(-3)^{k-1} + 2(-3)^k \\ &= \frac{1-(-3)^k}{2} + 2(-3)^k \\ &= \frac{1-(-3)^k + 4(-3)^k}{2} \\ &= \frac{1+3(-3)^k}{2} \\ &= \frac{1-(-3)^{k+1}}{2} \\ &= \text{RHS} \end{aligned}$$

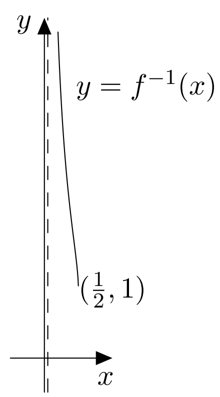
Therefore, by the principle of mathematical induction, the statement is true for all integers $n \geq 1$.

(b)

$$f(x) = \frac{x}{x^2+1}$$

(i) The domain of $f(x)$ is $x \geq 1$ as given in the question. The range is $0 < f(x) \leq \frac{1}{2}$ by inspection of the graph. Now for the domain and range of the inverse function, $f^{-1}(x)$, we switch the domain and range of $f(x)$. Hence the domain is $0 < x \leq \frac{1}{2}$ and the range is $y \geq 1$.

(ii)



(iii)
$$x = \frac{y}{y^2+1}$$
$$xy^2 + x = y$$
$$xy^2 - y + x = 0.$$

Using the quadratic formula, $\Delta = b^2 - 4ac$
$$= (-1)^2 - 4(x)(x)$$
$$= 1 - 4x^2.$$

This means for $f^{-1}(x)$, we have

$$y = \frac{1 \pm \sqrt{1-4x^2}}{2x}$$
$$f^{-1}(x) = \frac{1 \pm \sqrt{1-4x^2}}{2x}$$

Since $x > 1$ and $y \leq \frac{1}{2}$ for $f(x)$, then for the inverse function, $y > 1$ and $0 < x \leq \frac{1}{2}$

$$\therefore f^{-1}(x) = \frac{1 + \sqrt{1-4x^2}}{2x}.$$

(c) Consider the equations for displacement, velocity, and acceleration

$$\ddot{x} = 0 \qquad \ddot{y} = -g$$
$$\dot{x} = v\cos\theta \qquad \dot{y} = -gt + v\sin\theta$$
$$x = vt\cos\theta \qquad y = \frac{-gt^2}{2} + vt\sin\theta$$

(i) The range of a projectile is the x value when $y = 0$. Hence,

$$\frac{-gt^2}{2} + vt\sin\theta = 0$$
$$t = \frac{2v\sin\theta}{g}$$

Substituting into the $x(t)$ equation, we have $R = v\cos\theta\left[\frac{v\sin\theta}{g}\right]$
$$R = \frac{2v^2\sin\theta\cos\theta}{g}$$
$$\therefore R = \frac{v^2\sin 2\theta}{g}$$

(ii) Let $\theta_1 = \frac{\pi}{2} - \theta$.

$$R = \frac{v^2}{g}\sin(\pi - 2\theta) \qquad \text{(using supplementary angles)}$$
$$\therefore R = \frac{v^2}{g}\sin 2\theta$$

(iii) To find the maximum height, let the vertical velocity be equal to zero and sub the time into displacement, y. Note that maximum height occurs when there is no vertical velocity.

Letting $\dot{y} = 0, \quad -gt + v\sin\theta = 0$

$$\implies t = \frac{v\sin\theta}{g}$$

At $\theta = \alpha$, $t = \dfrac{v\sin\alpha}{g}$

At $\theta = \beta$, $t = \dfrac{v\sin\beta}{g}$

Maximum Height: $h = \dfrac{-gt^2}{2} + vt\sin\theta$

At $\theta = \alpha$, $h_\alpha = \dfrac{-gt^2}{2} + vt\sin\alpha$

$$\begin{aligned}\text{Hence, } h_\alpha &= -\frac{g}{2}\Big[\frac{v^2\sin^2\alpha}{g^2}\Big] + v\sin\alpha\Big[\frac{v\sin\alpha}{g}\Big] \\ &= \frac{-v^2\sin^2\alpha}{2g} + \frac{v^2\sin^2\alpha}{g} \\ &= \frac{v^2\sin^2\alpha}{2g}\end{aligned}$$

At $\theta = \beta$, $h_\beta = \dfrac{v^2\sin^2\beta}{2g}$

$$\begin{aligned}\text{Average Height} &= \frac{h_\alpha + h_\beta}{2} \\ &= \frac{\frac{v^2\sin^2\alpha}{2g} + \frac{v^2\sin^2\beta}{2g}}{2} \\ &= \frac{v^2}{4g}\Big[\sin^2\alpha + \sin^2\beta\Big]\end{aligned}$$

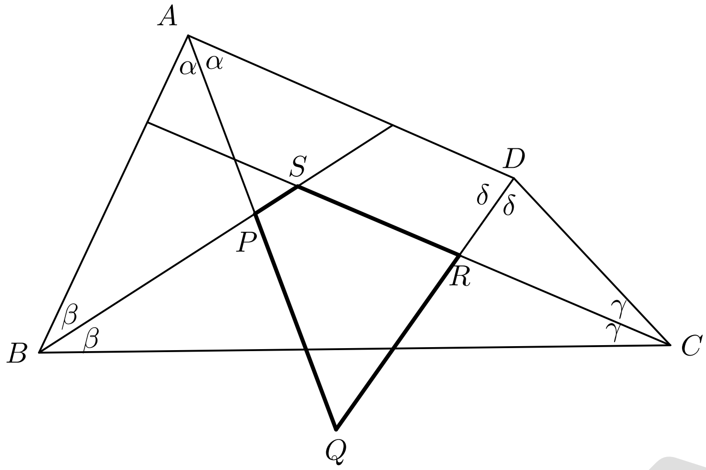
Since the question wants us prove that the average height is dependent only on velocity and gravity, this means that we must eliminate α and β. To do so, we need a relationship between α and β. From the question, we let $\beta = \dfrac{\pi}{2} - \alpha$.

$$\begin{aligned}\text{Average Height} &= \frac{v^2}{4g}\Big[\sin^2\alpha + \sin^2\left(\frac{\pi}{2} - \alpha\right)\Big] \\ &= \frac{v^2}{4g}\Big[\sin^2\alpha + \cos^2\alpha\Big] \\ &= \frac{v^2}{4g}\end{aligned}$$

$\therefore$ The average height depends only on v and g

Question 14

(a) Consider the diagram:



Method 1.

In quadrilateral $ABCD$,

$$\begin{aligned}\angle ABC + \angle BCD + \angle ADC + \angle BAD &= 360^\circ \text{ (angle sum of quad } ABCD \text{ is } 360^\circ)\\ 2\alpha + 2\beta + 2\gamma + 2\delta &= 360^\circ\\ \alpha + \beta + \gamma + \delta &= 180^\circ\end{aligned}$$

In $\triangle ADQ$,

$$\begin{aligned}\angle ADQ + \angle DAQ + \angle AQD &= 180^\circ \text{ (angle sum of } \triangle ADQ \text{ is } 180^\circ)\\ \angle PQR + \alpha + \delta &= 180^\circ \text{ (since } \angle PQR \text{ is } \angle AQD)\\ \angle PQR &= 180^\circ - \alpha - \delta\end{aligned}$$

Similarly in $\triangle BCS$, $\angle PSR = 180^\circ - \beta - \gamma$.

$$\begin{aligned}\angle PQR + \angle PSR &= 180^\circ - \alpha - \delta + 180^\circ - \beta - \gamma\\ &= 360^\circ - (\alpha + \beta + \gamma + \delta)\\ &= 360^\circ - 180^\circ\\ &= 180^\circ\end{aligned}$$

Since opposite angles in quadrilateral $PQRS$ are supplementary, $PQRS$ is a cyclic quadrilateral.

Method 2.

$$\begin{aligned}2\alpha + 2\beta + 2\gamma + 2\delta &= 360^\circ \text{ (angle sum of quadrilateral is } 360^\circ)\\ \alpha + \beta + \gamma + \delta &= 180^\circ\\ \angle DRC + \angle RCD + \angle CDR &= 180^\circ \text{ (angle sum of } \triangle RDC = 180^\circ)\\ \angle DRC + \gamma + \delta &= 180^\circ\\ \angle DRC &= 180^\circ - (\gamma + \delta)\\ &= \alpha + \beta \qquad (\alpha + \beta + \gamma + \delta = 180^\circ)\\ \angle SRQ &= \angle DRC \qquad \text{(vertically opposite angles are equal)}\\ &= \alpha + \beta\end{aligned}$$

Similarly, we can use angle sum of triangle APB to show that

$$\angle APB = 180^\circ - (\alpha + \beta)$$

and then use vertically opposite angles to deduce that

$$\begin{aligned}\angle SPQ &= \angle APB \\ &= 180^\circ - (\alpha + \beta)\end{aligned}$$

Hence

$$\begin{aligned}\angle SPQ + \angle SRQ &= (180^\circ - (\alpha + \beta)) + (\alpha + \beta) \\ &= 180^\circ\end{aligned}$$

This means that opposite angles of quadrilateral $PQRS$ are supplementary. For this reason, $PQRS$ must be a cyclic quadrilateral.

(b) (i) Note: $[1+(1+x)]^n \equiv (2+x)^n$

$$(2+x)^n = \binom{n}{0}2^n x^0 + \binom{n}{1}2^{n-1}x^1 + \binom{n}{2}2^{n-2}x^2 + \cdots + \binom{n}{n}2^0 x^n$$
$$[1+(1+x)]^n = \binom{n}{0}1^n(1+x)^o + \binom{n}{1}(1)^{n-1}(1+x)^1 + \cdots + \binom{n}{1}1^0(1+x)^n$$

Therefore

$$(2+x)^n = \sum_{k=0}^{n}\binom{n}{k}2^{n-k}x^k \cdots (1)$$
$$[1+(1+x)]^n = \sum_{k=0}^{n}\binom{n}{k}1^{n-k}(1+x)^k \cdots (2)$$

Note that $\binom{n}{r}2^{n-r}$, which is the RHS of the result, is the co-efficient of x^r in expansion (1). Consider the coefficient of x^r in equation (2).
Noting that

$$(1+x)^k = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n,$$

the smallest power of $(1+x)^k$ that contains an x^r term is $k = r$.

- Coefficient of x^r from $\binom{n}{r}(1+x)^r$ is $\binom{n}{r}\cdot\binom{r}{r}$
- Coefficient of x^r from $\binom{n}{r+1}(1+x)^{r+1}$ is $\binom{n}{r+1}\cdot\binom{r+1}{r}$
- Coefficient of x^r from $\binom{n}{r+2}(1+x)^{r+2}$ is $\binom{n}{r+2}\cdot\binom{r+2}{r}$

$\vdots \quad \vdots$

- Coefficient of x^r from $\binom{n}{n}(1+x)^n$ is $\binom{n}{n}\cdot\binom{n}{r}$

Hence coefficient of x^r from (2) is

$$\binom{n}{r}\binom{r}{r} + \binom{n}{r+1}\binom{r+1}{r} + \cdots + \binom{n}{n}\binom{n}{r}.$$

However, due to the equivalence of $(1+(1+x))^n$ and $(2+x)^n$, these coefficients of x^r from (1) and (2) must be equivalent.

$$\therefore \binom{n}{r}\binom{r}{r} + \binom{n}{r+1}\binom{r+1}{r} + \cdots + \binom{n}{n}\binom{n}{r} = \binom{n}{r}2^{n-r}$$

(ii) The group created by A could have a size of 4, 5, 6, ..., 23 people being chosen out of a pool of 23 people. For any group of a size of k which was formed by selector A, selector B needs to select 4 people out of selector A's pool of k people.
Hence the number of ways this selection process could be carried out is

$$\begin{aligned}\text{Ways} &= \binom{23}{4}\binom{4}{4} + \binom{23}{5}\binom{5}{4} + \binom{23}{6}\binom{6}{4} + \cdots + \binom{23}{23}\binom{23}{4}\\ &= \binom{23}{4}2^{23-4} \text{ (previous part with } n = 23, r = 4)\\ &= 4\,642\,570\,240.\end{aligned}$$

(c) (i) We are required to show that $\triangle ABC ||| \triangle ACD$.

In $\triangle ABC$ and $\triangle ACD$,

- $\angle ABC = \angle ADC = 90°$ (given $BC \perp AC$ and $OD \perp AB$)
- $\angle BAC = \angle CAD$ (common angle)

$\therefore \triangle ABC ||| \triangle ACD$ (equiangular)

(ii) $\dfrac{AB}{AC} = \dfrac{AC}{AD} = \dfrac{BC}{CD}$ (corresponding sides of similar triangles, $\triangle ABC$ and $\triangle ACD$, are in proportion

$$\therefore \frac{AB}{AC} = \frac{BC}{CD}$$
$$\frac{c}{b} = \frac{a}{x}$$
$$x = \frac{ab}{c}$$

(iii) In $\triangle ACD$ and $\triangle AFH$,

- $\angle ADC = \angle AHF$ (both 90 degrees, $CD \perp AB, FH \perp AB$)
- $\angle DAC = \angle HAF$ (common angle)

$\therefore \triangle ACD ||| \triangle AFH$ (equiangular)
Then since matching sides of similar triangles above are in proportion,

$$\frac{FH}{CD} = \frac{AF}{AC}$$
$$\frac{x_1}{x} = \frac{AC - CF}{AC}$$
$$= \frac{b - x}{b}$$
$$= \frac{b - \frac{ab}{c}}{b} \quad \text{(previous part)}$$
$$= \frac{c - a}{c}.$$
$$\therefore x_1 = x\left(\frac{c-a}{c}\right)$$

Likewise, we can show that $x_2 = x\left(\dfrac{c-a}{c}\right)^2$, $x_3 = x\left(\dfrac{c-a}{c}\right)^3$, and so on.

Hence the sum of the quadrants (quarter-circles) is

$$\begin{aligned}
\text{Area}_{\text{total}} &= \frac{\pi x^2}{4} + \frac{\pi x_1^2}{4} + \frac{\pi x_2^2}{4} + \cdots \\
&= \frac{\pi}{4}\left(x^2 + \left(x\left(\frac{c-a}{c}\right)\right)^2 + \left(x\left(\frac{c-a}{c}\right)^2\right)^2 + \cdots\right) \\
&= \frac{\pi}{4} \cdot \frac{x^2}{1-\left(\frac{c-a}{c}\right)^2} \qquad \text{(since the sum converges: see below)} \\
&= \frac{\pi}{4} \cdot \frac{\left(\frac{ab}{c}\right)^2}{1-\left(\frac{c-a}{c}\right)^2} \\
&= \frac{\pi}{4} \cdot \frac{\frac{a^2b^2}{c^2}}{\frac{c^2-(c^2-2ac+a^2)}{c^2}} \\
&= \frac{\pi}{4} \cdot \frac{a^2b^2}{2ac-a^2} \\
&= \frac{\pi ab^2}{4(2c-a)}.
\end{aligned}$$

Note that the limiting sum converges as

$$\begin{aligned}
|r| &= \left(\frac{c-a}{c}\right)^2 \\
&= \left(1-\frac{a}{c}\right)^2 \\
&< (1)^2 \qquad \text{(as } 0 < a < c\text{)} \\
&= 1.
\end{aligned}$$

(iv) Area of $\triangle ABC = \frac{1}{2}ab$

Area of quadrants $<$ Area of $\triangle ABC$

$$\begin{aligned}
\frac{\pi ab^2}{4(2c-a)} &< \frac{1}{2}ab \\
\frac{2\pi}{4} &< \frac{ab(2c-a)}{ab^2} \qquad \text{(as } a, b, c > 0 \text{ and } c > a\text{, so } 2c-a > 0\text{)} \\
\frac{\pi}{2} &< \frac{2c-a}{b}.
\end{aligned}$$